

THE DE SITTER/ANTI- DE SITTER BLACK HOLES PHASE TRANSITION?

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Abstract

We investigate the Schwarzschild-Anti-deSitter (SAdS) and SdS BH thermodynamics in 5d higher derivative gravity. The interesting feature of higher derivative gravity is the possibility for negative (or zero) SdS (or SAdS) BH entropy which depends on the parameters of higher derivative terms. The appearance of negative entropy may indicate a new type instability where a transition between SdS (SAdS) BH with negative entropy to SAdS (SdS) BH with positive entropy would occur or where definition of entropy should be modified.

Keywords: Black hole thermodynamics, (anti-)de Sitter space, higher derivative gravity

BH thermodynamics is quite attractive, as it provides the understanding of gravitational physics at extremal conditions. Moreover, it has been realized recently that AdS BH may be relevant in the study of AdS/CFT correspondence. Hence, there appears nice way to describe strong coupling gauge theories via their gravitational duals. In the present work we discuss the thermodynamics of dS and AdS BHs in higher derivative gravity. The fundamental issue of entropy for such objects leads to some interesting conclusions.

We start from the following action of d dimensional R^2 -gravity with cosmological constant. The action is given by:

$$S = \int d^{d+1}x \sqrt{-g} \left\{ aR^2 + bR_{\mu\nu}R^{\mu\nu} + cR_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma} + \frac{1}{\kappa^2}R - \Lambda \right\} . \quad (1)$$

We discuss the relation between SdS and SAdS BHs based on entropy considerations. For simplicity, we consider $c = 0$ case in (1) for most results. When $c = 0$, Schwarzschild-anti de Sitter space is an exact solution:

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{-2\nu(r)}dr^2 + r^2 \sum_{i,j=1}^{d-1} \tilde{g}_{ij}dx^i dx^j ,$$

$$e^{2\nu} = e^{2\nu_0} \equiv \frac{1}{r^{d-2}} \left(-\mu + \frac{kr^{d-2}}{d-2} + \frac{r^d}{l^2} \right) . \quad (2)$$

Here \tilde{g}_{ij} is the metric of the Einstein manifold, which is defined by $\tilde{R}_{ij} = kg_{ij}$. \tilde{R}_{ij} is the Ricci curvature given by \tilde{g}_{ij} and k is a constant. For example, one has $k = d - 2$ for $d - 1$ -dimensional unit sphere, $k = -(d - 2)$ for $d - 1$ -dimensional unit hyperboloid, and $k = 0$ for flat surface. The curvatures have the following form: $\hat{R} = -\frac{d(d+1)}{l^2}$ and $\hat{R}_{\mu\nu} = -\frac{d}{l^2}\hat{G}_{\mu\nu}$. In (1), μ is the parameter corresponding to mass and the scale parameter l is given by solving the following equation:

$$0 = \frac{d^2(d+1)(d-3)a}{l^4} + \frac{d^2(d-3)b}{l^4} - \frac{d(d-1)}{\kappa^2 l^2} - \Lambda . \quad (3)$$

When the solution for l^2 (3) is positive (negative), the spacetime is asymptotically anti-de Sitter (de Sitter) space and especially if $\mu = 0$, the solution expresses the pure anti-de Sitter or de Sitter space. If $\Lambda = 0$, there is asymptotically flat (Minkowski) solution, where $\frac{1}{l^2} = 0$. In the following we concentrate on the case of $d = 4$.

The calculation of thermodynamical quantites like free energy F , the entropy \mathcal{S} and the energy E may be done with the help of the following method: After Wick-rotating the time variable by $t \rightarrow i\tau$, the free energy F can be obtained from the action S (1) where the classical solution is substituted : $F = -TS$. Substituting eq.(3) into (1) in the case of $d = 4$ with $c = 0$, one gets

$$S = - \int d^5x \sqrt{-G} \left(\frac{8}{l^2 \kappa^2} - \frac{320a}{l^4} - \frac{64b}{l^4} \right)$$

$$= - \frac{V_3}{T} \int_{r_H}^{\infty} dr r^3 \left(\frac{8}{l^2 \kappa^2} - \frac{320a}{l^4} - \frac{64b}{l^4} \right) \quad (4)$$

Here V_3 is the volume of 3d sphere and we assume τ has a period of $\frac{1}{T}$. The expression of S contains the divergence coming from large r . In order to subtract the divergence, we regularize S (3) by cutting off the integral at a large radius r_m and subtracting the solution with $\mu = 0$:

$$S_{\text{reg}} = -\frac{V_3}{T} \left\{ \int_{r_H}^{r_m} dr r^3 - \frac{e^{\nu(r=r_m)}}{e^{\nu(r=r_m;\mu=0)}} \int_0^{r_m} dr r^3 \right\} \left(\frac{8}{\kappa^2 l^2} - \frac{320a}{l^4} - \frac{64b}{l^4} \right) \quad (5)$$

The factor $e^{\nu(r=r_m)}/e^{\nu(r=r_m;\mu=0)}$ is chosen so that the proper length of the circle which corresponds to the period $\frac{1}{T}$ in the Euclidean time at $r = r_m$ coincides with each other in the two solutions. Taking $r_m \rightarrow \infty$, one finds, as found in [1],

$$F = -V_3 \left(\frac{l^2 \mu}{8} - \frac{r_H^4}{4} \right) \left(\frac{8}{l^2 \kappa^2} - \frac{320a}{l^4} - \frac{64b}{l^4} \right) \quad (6)$$

The horizon radius r_h is given by solving the equation $e^{2\nu_0(r_H)} = 0$ in (1):

$$r_H^2 = -\frac{kl^2}{4} + \frac{1}{2} \sqrt{\frac{k^2}{4} l^4 + 4\mu l^2} . \quad (7)$$

The Hawking temperature T_H is

$$T_H = \frac{k}{4\pi r_H} + \frac{r_H}{\pi l^2} \quad (8)$$

where $'$ denotes the derivative with respect to r . From the above equation (8), r_H can be rewritten in terms of T_H as

$$r_H = \frac{1}{2} \left(\pi l^2 T_H \pm \sqrt{(\pi l^2 T_H)^2 - k l^2} \right) \quad (9)$$

In (9), the plus sign corresponds to $k = -2$ or $k = 0$ case and the minus sign to $k = 2$ case.¹ One can also rewrite μ by using r_H as $\mu = \frac{r_H^4}{l^2} + \frac{kr_H^2}{2}$. Then we can rewrite F using r_H as

$$F = -\frac{V_3}{8} r_H^2 \left(\frac{r_H^2}{l^2} - \frac{k}{2} \right) \left(\frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right) . \quad (10)$$

Then the entropy $\mathcal{S} = -\frac{dF}{dT_H}$ and the thermodynamical energy $E = F + T\mathcal{S}$ can be obtained as follows [1]:

$$\mathcal{S} = 4V_3 \pi r_H^3 \left(\frac{1}{\kappa^2} - \frac{40a}{l^2} - \frac{8b}{l^2} \right) , \quad E = 3V_3 \mu \left(\frac{1}{\kappa^2} - \frac{40a}{l^2} - \frac{8b}{l^2} \right) , \quad (11)$$

This seems to indicate that the contribution from the R^2 -terms can be absorbed into the redefinition:

$$\frac{1}{\tilde{\kappa}^2} = \frac{1}{\kappa^2} - \frac{40a}{l^2} - \frac{8b}{l^2} , \quad (12)$$

although this is not true for $c \neq 0$ case.

On the other hand, by using the surface counter term method [2] , one gets the following expression for the conserved mass M :

$$M = \frac{3l^2}{16} V_3 \left(\frac{1}{\kappa^2} - \frac{40a}{l^2} - \frac{8b}{l^2} - \frac{4c}{l^2} \right) \left(k^2 + \frac{16\mu}{l^2} \right) . \quad (13)$$

One can also start from the expression for M with $c = 0$ as the thermodynamical energy E :

$$E = 3V_3 \left(\frac{1}{\kappa^2} - \frac{40a}{l^2} - \frac{8b}{l^2} \right) \left(\frac{k^2 l^2}{16} + \frac{\mu}{l^2} \right) \quad (14)$$

The expression of energy E (14) is different from that in (11) by a first μ -independent term, which comes from the AdS background. Using the thermodynamical relation $d\mathcal{S} = \frac{dE}{T}$, we find

$$\mathcal{S} = \int \frac{dE}{T_H} = \int dr_H \frac{dE}{d\mu} \frac{d\mu}{dr_H} \frac{1}{T_H} = \frac{V_3 \pi r_H^3}{2} \left(\frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right) + \mathcal{S}_0 . \quad (15)$$

Here \mathcal{S}_0 is a constant of the integration. Up to the constant \mathcal{S}_0 , the expression (15) is identical with (11). We should note that the entropy \mathcal{S} (11) becomes negative, when

$$\frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} < 0 . \quad (16)$$

This is true even for the expression (15) for the black hole with large radius r_H since \mathcal{S}_0 can be neglected for the large r_H .

We now investigate in more detail what happens when Eq.(16) is satisfied. First we should note l^2 is determined by (3), which has, in case of $d = 4$, the following form:

$$0 = \frac{80a + 16b}{l^4} - \frac{12}{\kappa^2 l^2} - \Lambda , \quad (17)$$

There are two real solutions for l^2 when $\frac{6}{\kappa^2} + (80a + 16b) \Lambda \geq 0$ and the solutions are given by

$$\frac{1}{l^2} = \frac{\frac{6}{\kappa^2} \pm \sqrt{\frac{6}{\kappa^2} + (80a + 16b) \Lambda}}{80a + 16b} . \quad (18)$$

Suppose $\kappa^2 > 0$. Then if

$$(80a + 16b) \Lambda > 0 , \quad (19)$$

one solution is positive but another is negative. Therefore there are both of the asymptotically AdS solution and asymptotically dS one. Let us denote the positive solution for l^2 by l_{AdS}^2 and the negative one by $-l_{\text{dS}}^2$:

$$l^2 = l_{\text{AdS}}^2, -l_{\text{dS}}^2, \quad l_{\text{AdS}}^2, l_{\text{dS}}^2 > 0 . \quad (20)$$

Then when the asymptotically AdS solution is chosen, the entropy (15) has the following form:

$$\mathcal{S}_{\text{AdS}} = \frac{V_3 \pi r_H^3}{2} \left(\frac{8}{\kappa^2} - \frac{320a + 64b}{l_{\text{AdS}}^2} \right) . \quad (21)$$

Here we have chosen $\mathcal{S}_0 = 0$. On the other hand, when the solution is asymptotically dS, the entropy (15) has the following form:

$$\mathcal{S}_{\text{dS}} = \frac{V_3 \pi r_H^3}{2} \left(\frac{8}{\kappa^2} + \frac{320a + 64b}{l_{\text{dS}}^2} \right) . \quad (22)$$

When

$$\frac{8}{\kappa^2} - \frac{320a + 64b}{l_{\text{AdS}}^2} < 0 , \quad (23)$$

the entropy \mathcal{S}_{AdS} (21) is negative!

There are different points of view to this situation. Naively, one can assume that above condition is just the equation to remove the non-physical domain of theory parameters. However, it is difficult to justify such proposal. Why for classical action on some specific background there are parameters values which are not permitted? Moreover, the string/M-theory and its compactification would tell us what are the values of the theory parameters.

From another side, one can conjecture that classical thermodynamics is not applied here and negative entropy simply indicates to new type of instability in asymptotically AdS black hole physics. Indeed, when Eq.(23) is satisfied, since $80a + 16b > 0$ (same range of parameters!), the entropy \mathcal{S}_{dS} (22) for asymptotically dS solution is positive. In other words, may be the asymptotically dS solution would be preferable?

On the other hand, when

$$\frac{8}{\kappa^2} + \frac{320a + 64b}{l_{\text{dS}}^2} < 0 , \quad (24)$$

the entropy \mathcal{S}_{dS} in (22) is negative and the asymptotically dS solution is instable (or does not exist). In this case, since $80a + 16b < 0$, the entropy \mathcal{S}_{AdS} in (21) for asymptotically AdS solution is positive and the asymptotically AdS solution would be preferable. Expression for the AdS black hole mass in (14) tells that when $\frac{8}{\kappa^2} - \frac{320a+64b}{l_{\text{AdS}}^2} = 0$, the AdS black hole becomes massless then there would occur the condensation of the black holes, which would make the transition to the dS black hole. On the other hand, when $\frac{8}{\kappa^2} + \frac{320a+64b}{l_{\text{dS}}^2} = 0$, the dS black hole becomes massless then there would occur the condensation of the black holes and the AdS black hole would be produced. Note that above state with zero entropy (and also zero free energy and zero conserved BH mass) is very interesting. Perhaps, this is some new state of BHs. As we saw that is this state which defines the border between physical SAdS (SdS) BH with positive entropy and SdS (SAdS) BH with negative entropy.

Hence, there appeared some indication that some new type of phase transition (or phase transmutation) between SdS and SAdS BHs in higher derivative gravity occurs. Unfortunately, we cannot suggest any dynamical formulation to describe explicitly such phase transition (it is definitely phase transition not in standard thermodynamic sense).

Let us consider now the entropy for Gauss-Bonnet case, where $a = c$ and $b = -4c$ in (1). For this purpose, we use the thermodynamical relation $d\mathcal{S} = \frac{dE}{T}$. For the Gauss-Bonnet case, the energy (13) has the following form [2]:

$$E = M = \frac{3l^2}{16} V_3 \left(\frac{1}{\kappa^2} - \frac{12c}{l^2} \right) \left(k^2 + \frac{16\mu}{l^2} \right) \quad (25)$$

We also found [2]

$$\mu = \frac{1}{2l^2} \left(k\epsilon - \frac{1}{2} \right)^{-1} \left\{ (2\epsilon - 1)r_H^4 - kr_H^2 l^2 \right\} . \quad (26)$$

Here $\epsilon \equiv \frac{c\kappa^2}{l^2}$. Then using (25), (26), and the expression of the Hawking temperature,

$$4\pi T_H = \frac{1}{2} \left(ck + \frac{r_H^2}{2\kappa^2} \right)^{-1} \left[\frac{kr_H}{\kappa^2} - \frac{8cr_H^3}{l^4} + \frac{4r_H^3}{\kappa^2 l^2} - \frac{2Q^2}{3g^2 r_H^3} \right] , \quad (27)$$

the entropy can be obtained as

$$\mathcal{S} = \int \frac{dE}{T_H} = \int dr_H \frac{dE}{d\mu} \frac{\partial \mu}{\partial r_H} \frac{1}{T_H} \quad (28)$$

$$= \frac{V_3}{\kappa^2} \left(\frac{1 - 12\epsilon}{1 - 4\epsilon} \right) \left(4\pi r_H^3 + 24\epsilon k \pi r_H \right) + \mathcal{S}_0 . \quad (29)$$

Here \mathcal{S}_0 is a constant of the integration, which could be chosen to be zero if we assume $\mathcal{S} = 0$ when $r_H = 0$. When $\epsilon = 0$ ($c = 0$), the expression reproduces the standard result

$$\mathcal{S} \rightarrow \frac{4\pi V_3 r_H^3}{\kappa^2} . \quad (30)$$

The entropy (28) becomes negative (at least for the large black hole even if $\mathcal{S}_0 \neq 0$) when

$$\frac{1}{12} < \epsilon < \frac{1}{4} . \quad (31)$$

Therefore the unitarity might be broken in this region but it might be recovered when $\epsilon > \frac{1}{4}$. Even in case $\epsilon < 0$ ($k = 2$), the entropy becomes negative when

$$r_H^2 < -12\epsilon , \quad (32)$$

if $\mathcal{S}_0 = 0$. Then the small black hole might be unphysical.

The fact discovered here-that entropy for S(A)dS BHS in gravity with higher derivatives terms may be easily done to be negative by the corresponding choice of parameters is quite remarkable. It is likely that thermodynamics for black holes with negative entropies should be reconsidered. In this respect one possibility would be to redefine the gravitational entropy for higher derivative gravity.

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Notes

1. When $k = 2$, as we can see from (7) and (8), r_H , and also T_H , are finite in the limit of $l \rightarrow \infty$, which corresponds to the flat background. Therefore we need to choose the minus sign in (9) for $k = 2$ case.

References

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